# LANCHESTER'S THEORY OF COMBAT: THE STATE OF THE ART IN MID-1970

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# THESIS

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ABSTRACT

Since its introduction more than half a century ago, Lanchester's theory of combat has undergone considerable revision and expansion. This thesis presents a consolidation and a grouping by subject of the significant contributions to this theory which have appeared in the literature in recent years, with emphasis on the period since 1964, when a thorough "state of the art" summary article was published by Dolansky. The important results of each notable effort are described.

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# I. INTRODUCTION AND CATEGORIZATION OF REFERENCES

Dynamic combat was first modeled explicity by F. W. Lanchester [39] (although his work was anticipated by Rear Admiral Bradley A. Fiske [20, 62]). Lanchester formulated a battle between two forces in terms of a pair of simultaneous differential equations which described the rate at which each side's strength would be decreased. While his model emerged as a significant contribution to a new field, it had limited applicability because it was based on several simplifying but restricting assumptions. In the years that have passed since the publication of Lanchester's work, and particularly since the early 1940's when operations research came to the forefront as a tool for the military decision-maker, much has been written on the subject of combat dynamics. Some authors have challenged the applicability of the Lanchester equations under any circumstances. Others have modified the assumptions behind the original equations in order to use them as a predictive device for present and possible future combat situations, to include ambushes, guerrilla warfare, and the like. Efforts have been made to determine from a given set of initial conditions the probability that one side or the other will ultimately win. Additionally, the original deterministic combat process has been formulated as a stochastic process.

The purpose of this thesis is to review the literature on Lanchester's theory of combat in an attempt to assemble in

one document, by subject, the numerous extensions of and contributions to this theory which have been made through the years. Emphasis is on the period since 1964, at which time an extensive review was made by Dolansky [18].

Section II describes briefly Lanchester's original formulation. In Section III, most of the significant contributions to Lanchester's theory uncovered during the research for this thesis are described. Section IV contains a summary and a look at the future.

Two points must be kept in mind by the reader. First, this thesis does not attempt to describe in detail every article on Lanchester's theory which has ever appeared in the literature. For example, work by Morse and Kimball, while extensive, has been discussed by many authors over the years in numerous articles. All these articles, and the original works upon which they were based, are listed in the bibliography. Secondly, to avoid repetition, most of the detailed descriptions which appear on the following pages are of work which has been done since Dolansky's summary article was published. As can be seen by a quick glance at the bibliography, this six-year period represents a considerable number of articles. It is hoped that with this thesis and Dolansky's 1964 article in front of him, the interested analyst will have an accurate picture of the state of the art of Lanchester's theory of combat through mid-1970.

A grouping of the articles by subject appears below as a convenience to the reader. Many of the articles referenced

contain extensive bibliographies of their own, providing more complete information on the subject under consideration.

| Subject | Pertinent References |
|---|---|
| Amphibious Assaults | 35 |
| Attrition Coefficients | 3, 5, 6, 7, 16, 21, 45 |
| Before Lanchester | 20, 62 |
| Command Efficiency and Intelligence | 49 |
| Duels | 5, 25 |
| Force Sizes Grossly Unequal | 26 |
| General Theory and Summary Articles | 1, 2, 4, 5, 8, 9, 18, 23, 37, 42, 55, 57, 59, 60 |
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## II. THE ORIGINAL LANCHESTER FORMULATION

Lanchester's equations, now in existence for more than 50 years, represent a significant contribution to the art of modeling the dynamics of military combat. Lanchester [39] described the attrition of each side in a two-sided struggle by means of a system of ordinary differential equations. Beginning with three basic assumptions, he postulated two types of attrition: the linear law and the square law. The assumptions common to both theories were:

1. Two military forces (red and blue) are opposing each other. On each side, every soldier is armed with the same weapon. The attrition rate at which a single weapon of one kills units of the other side may not be the same for each side.

2. Every weapon on each side can take under fire all weapons of the opposing side.

3. The attrition rates for each side are known and do not change for the duration of the engagement.

The linear law results under the circumstances where each side is ignorant of the exact location of its opposition but does have knowledge of the general area occupied (area fire). Furthermore, as units of each side are destroyed, the survivors distribute their fire uniformly over the area occupied by the surviving opponents. With these additional assumptions, Lanchester's linear law may be written as follows:

$$\frac{dR}{dt} = -A_{br}B(t)R(t),$$

and

$$\frac{dB}{dt} = -A_{rb}R(t)B(t),$$

where $\frac{dR}{dt}$ is the rate at which the red force decreases in strength, $A_{br}$ is the constant attrition rate for force blue (the rate at which a single blue unit kills red units), and B(t) and R(t) are the number of surviving blue and red units, respectively, at time t. Similarly, $\frac{dB}{dt}$ is the rate at which the blue force decreases in strength and $A_{rb}$ is the rate at which a single red unit kills blue units.

Lanchester's square law is applicable in the situation where each unit of both sides knows the precise location of all surviving units of its enemy, so that as opponents are eliminated, fire is immediately shifted to and uniformly distributed over all surviving units (aimed fire). With this scenario, Lanchester's equations may be written:

$$\frac{dR}{dt} = -A_{br}B(t),$$

and

$$\frac{dB}{dt} = -A_{rb}R(t),$$

where the symbols are as defined above.

If one accepts Lanchester's assumptions, his equations are most valuable. From knowledge of the type fire employed by each side (aimed fire or area fire), the strength of each

side, and the rate at which each side inflicts casualties, one can determine each side's strength at any time t by solving two simultaneous differential equations.

# III. RECENT CONTRIBUTIONS TO LANCHESTER'S THEORY

## A. GENERAL

In recent years a considerable amount of research has been done in Lanchester's theory, much of it obviously inspired by suggestions from Dolansky [18]. In this chapter, this work is summarized, by category, in order to bring the reader up-to-date on the current state of the art through mid-1970.

## B. HETEROGENEOUS FORCES

Lanchester's assumption of the homogeneity of forces on each side is obviously not appropriate in modern warfare. Seldom, if ever, again will a force operate on any battlefield with only one type weapon. Work on the incorporation into the model of different weapon types on each side is not new (see Helmer [32], Snow [53], and Weiss [59]). Snow [53] criticized those who have attempted to fit the original Lanchester equations to combat between mixed forces by assuming a linear form. He pointed out that the assumption of a linear form prohibits mutual support within each force. Clearly, mutual support will be present, as, for example, when one side is composed of tanks and infantry. A recent contribution which considered heterogeneous forces was made by Helmbold [30]. Two heterogeneous forces opposed each other in a series of volleys. Every firing unit on each side, regardless of type, followed a predetermined plan of attack and did not adjust

its behavior as the situation changed. Helmbold assumed that the probability that a given fire unit on one side survived fire from a given fire unit of the other side was independent of what all other fire units of the other side did. Since each side followed a predetermined attack plan (rather than changing its tactics as casualties were suffered by both sides), the model was less valid in the later stages of battle. With the progress of the battle described as above, Helmbold derived the following equations for the expected number of weapons surviving each volley:

$$E[R_j] = \sum_{r \varepsilon R} E[S_r^j] = \sum_{r \varepsilon R} Q_r^j ,$$

and

$$E[B_j] = \sum_{b \varepsilon B} E[S_b^j] = \sum_{b \varepsilon B} P_b^j .$$

In the above equations, $E[R_j]$ and $E[B_j]$ represent the expected number of weapons on the red and blue sides, respectively, which survive volley j; $S_r^j$ is defined to be one if fire unit r survives volley j, and zero, otherwise (similarly for $S_b^j$); $Q_r^j$ is defined as the probability that $S_r^j = 1$ (similarly for $P_b^j$). $Q_r^j$ is calculated from the following formula: $Q_r^j = Q_r^{j-1} \prod_{b \varepsilon B} (1-q_{br}^j)$, where $q_{br}^j$ = the probability that fire unit $b \varepsilon B$ kills fire unit $r \varepsilon R$ during volley j. (An analogous expression existed for $P_b^j$.) While the results which Helmbold's equations yielded would be invaluable to any commander, these results depended entirely on a set of probabilities which were often difficult to accurately predict.

## C. STOCHASTIC FORMULATIONS

Perhaps the most extensive work done with Lanchester's equations has centered on the reformulation of the problem in stochastic form. The original Lanchester equations were deterministic; that is, they predicted annihilation of one side or the other at some precise moment in the future, and yielded the same results every time for the same parameters. Clearly, such an approach was not in accordance with reality. But, Lanchester's deterministic equations could be considered as an expected value formulation of combat, and thus could be derived by taking expected values of the underlying probability distributions. Early work in stochastic formulations was done by B. O. Koopman [37] during World War II. Other pioneers in this field were Snow [53] and Morse and Kimball [43]. In the paragraphs which follow, several more recent stochastic formulations are described. Each of the authors whose work is outlined assumed that combat proceeds according to a Markov process (no dependence on past history) and that the process is stationary (that which happens during any time interval from $t$ to $t+\Delta t$ depends upon the state of the two forces at time $t$, and upon the time length, $\Delta t$, but is independent of the time, $t$, at which the interval begins).

Brown [12] described two-sided combat as a Markov stochastic process characterized by the initial state of the system (strengths of each side) and two functions, $\phi$ and $\alpha$.

He assumed that a system in state (r,b) at the beginning of any short time interval of length t could do one of three things during that time interval. It could remain in state (r,b) (no casualties on either side); it could go to (r,b-1) (one casualty on the blue side and none on the red side); or it could go to (r-1,b) (one casualty on the red side and none on the blue side). Thus, the time intervals were chosen to be short enough such that it was impossible for either side to lose more than one man or for each side to lose one man in any interval. In Brown's model, $e^{-t/\tau(r,b)}$ was used as the probability that the system remains in state (r,b) throughout a time interval of length t, where $\tau(r,b)$ is the expected duration of the system in state (r,b). $\phi$ was defined as $1/\tau(r,b)$. $\alpha(r,b)$ was taken as the probability that the system goes from (r,b) to (r,b-1) in one $\Delta t$ time interval, while $\beta(r,b)$ was the probability that the system goes from (r,b) to (r-1,b) in one $\Delta t$ time interval.

Two questions which a stochastic formulation of combat usually addresses are the probability that a particular side will win the battle and the probability that, given some initial state (r,b), the system will be in some other state $(r_1,b_1)$ after some time T. Brown showed that the probability that the red side wins, P(r,b) satisfies the difference equation,

$$P(r,b) = \alpha(r,b)P(r,b-1) + \beta(r,b)P(r-1,b).$$

He went on to show that

$$P(r_1,b_1,T;r,b) = \sum_{k\varepsilon I_{r,b}} P_k(r_1,b_1;r,b)\ G_k(r_1,b_1,T;r,b),$$

where $I_{r,b}$ is the set of feasible transition paths, $P_k(r_1,b_1;r,b)$ is the probability that the system goes from $(r,b)$ to $(r_1,b_1)$ along path k, and $G_k(r_1,b_1,T;r,b)$ is the probability that if a system travels path k, it will be at $(r_1,b_1)$ at time T if it started at (r,b). As was pointed out in the paper, $G_k$ is very difficult to compute if r+b is at all large. Brown did not attempt an exact solution to the difference equation for P(r,b), but he did derive a close approximation to a solution. It is wise to state explicitly before considering the next approach to stochastic Lanchester theory that fundamental to Brown's work was the assumption that r(t) and b(t), the number of survivors on each side at time t, were random variables, not completely determined beforehand.

Another stochastic model which used reasoning very similar to that employed by Brown was developed by Bram [10]. Although he used a different scenario, Bram's model differed from the one described above only in that Bram permitted a transition from (r,b) to (r-1,b-1) to occur with positive probability. In Bram's model, two naval forces opposed each other. On one side were r aircraft carriers (supported by escort forces and general ASW forces). On the other side were b submarines. The carriers were in an area attempting to accomplish a separate mission (search rate was $S_s$). The submarines had the mission of seeking and destroying the carriers (search rate

was $S_c$). At time t there were i carriers and j submarines in the area of operations. The model assumed that submarines could kill only carriers and that submarines were killed by escort forces and general ASW forces. The question Bram addressed was the determination of P(i,j,t;r,b). He defined c = 0 if the carrier was not killed in an encounter, and c = 1 if the carrier was killed (similarly, s = 0 and s = 1 for similar circumstances for the submarine). From these definitions, four probabilities, $P_{00}$, $P_{01}$, $P_{10}$, and $P_{11}$ could occur as possible values for $P_{cs}$, where, for example, $P_{01}$ was the probability that in an encounter, the carrier survived and the submarine was killed. $K_g$ was defined as the probability that a submarine was killed by ASW, given that it was detected by ASW. With these assumptions, Bram derived expressions for each of the four possible transition probabilities. In the interest of brevity, only the first of these will be described in detail. If the system was in state (i,j+1) at time t and in state (i,j) at time t+Δt, that implied that a submarine was killed. This could have happened in two ways: either the submarine was detected and killed by ASW (which occurred with probability $S_sK_g\Delta t(j+1)$), or there was an encounter with a carrier in which the carrier survived and the submarine was killed (which occurred with probability $S_c(i)(j+1)\Delta tP_{01}$). Thus,

$$P(i,j,t+\Delta t;i,j+1) = S_sK_g\Delta t(j+1) + S_c(i)(j+1)\Delta tP_{01} \, .$$

Similarly,

$$P(i,j,t+\Delta t;i+1,j) = S_c(i+1)(j)\Delta t P_{10} ,$$

$$P(i,j,t+\Delta t;i,j) = [1-S_sK_g(j)\Delta t][1-S_c\Delta t(i)(j)\cdot (P_{01}+P_{10}+P_{11})] ,$$

and

$$P(i,j,t+\Delta t;i+1,j+1) = S_c(i+1)(j+1)\Delta t P_{11} .$$

Since these four transition probabilities are mutually exclusive and exhaustive, $P(i,j,t+\Delta t)$ is equal to their sum. Bram showed that one could solve for $P(i,j,t+\Delta t)$ by taking its derivative with respect to time and specifying the initial conditions that $P_{r,b}(0) = 1$ and $P_{i,j}(0) = 0$. From its solution he calculated $E[i(t)]$, the expected number of red survivors at time t as being equal to $\sum_i \sum_j iP_{ij}(t)$ (similarly for $E[j(t)]$). In order to solve for $q(t)$, the probability that a carrier survives to time t, Bram noted that $q(t)$ is exactly equal to $\frac{1}{r} E[i(t)]$. He then developed two approximate exponential forms for $q(t)$, both of which were shown to be good approximations for the four cases examined. These expressions are:

$$q(t) \simeq \exp\left\{-\left[\frac{b(D-A)-0.0568r(D-B)}{K+0.6222r(D-B)}\right]\left[1-e^{-t(k+0.6222r(D-B)}\right]\right\},$$

and

$$q(t) \simeq \exp\left[\frac{-(D-A)}{K} \quad b(1-e^{-kt})\right],$$

where

$$A = S_c P_{01},\ B = S_c P_{10},\ C = S_c P_{11},\ D = A+B+C, \text{ and}$$

$$K = S_s K_g \ .$$

Robertson [48] developed a scenario for a probabilistic approach to Lanchester's theory which had eight defenders opposing nine attackers. The model did not allow for intervisibility. On each of a series of volleys, every defender chose an attacker at random. The eight defenders fired simultaneously (eight shots per volley). At the conclusion of each volley there existed a set of probabilities that 0,1,...,9 attackers had survived that volley. For example, the probability that five attackers survived a volley is the probability that four shots hit four attackers + the probability that five shots hit four attackers +...+ the probability that eight shots hit four attackers (Robertson assumed that an attacker was killed when hit). These sets of probabilities, when grouped as a sequence, formed a Markov chain which was the basis of weapon comparisons. With the Markovian nature of the process having been demonstrated, Robertson gave expressions for the one-step transition probabilities:

$P_i^j\ (S_n, n) = P(t_n = i \mid t_{n-1} = j \mid S_n$ shots with hit probabilities $P_k^n$, $k = 1,2,\ldots,S_n$, are fired on the $n^{th}$ volley), where $t_n$ is the number of attackers surviving the $n^{th}$ volley. The above expression applied only in the case where the hit probability was the same for all shots.

Helmbold [27] extended Robertson's basic model and put it in a more general framework to allow for intervisibility effects and differences between weapons. He then developed a series of discrete approximations to the expected number of attackers surviving a given volley. To allow for intervisibility limitations, $V_m^j(n)$ was defined as the probability that m targets are visible at the start of the $n^{th}$ volley, given that j targets are alive at the start of that volley. Then,

$$\hat{P}_i^j(s,n) = \sum_{m=0}^{j} P_{i-(j-m)}^m(s,n) V_m^j(n),$$

where $P_{i-(j-m)}^m(s,n)\ V_m^j(n)$ is the probability that (j-m) targets are not visible (and must, therefore, survive that volley) multiplied by the probability that [i-(j-m)] of the visible targets survive that volley (for a total of i surviving targets). Helmbold's expression for $\hat{P}_i^j(s,n)$, with intervisibility considered, was thus analogous to Robertson's transition probability formula when intervisibility was ignored. Helmbold went on to point out the inefficient results which would arise if the defenders fired at only a subset of the attacking force, thereby increasing the chances of multiple hits on a single target.

In the material which follows on the determination of the expected number of attackers surviving a given volley, it was assumed that all attackers are visible to the defenders.

Prior to deriving by several methods the approximate expected number of attackers surviving a given volley, Helmbold

pointed out that this number could be computed exactly from the distribution of $t_n$, but that such calculations were not easily done. His first approximation was a modification of an approximation which Robertson used. Robertson had assumed that the defenders' fire on each volley is uniformly distributed over the original number of attackers. As was reasonable, Helmbold argued that since the defenders see all surviving attackers, fire will be distributed only over those attackers who have not been killed. He showed that $E[t_{n+1}]$ (hereafter called $T_{n+1}$) could be approximated by

$$T_n \prod_{k=1}^{s} (1-p_k^n)^{1/T_n},$$

where $p_k^n$ = the probability that the $k^{th}$ shot on the $n^{th}$ volley is a hit. In the special case where each volley consisted of a single shot,

$$T_n \simeq T_o - \sum_{r=1}^{n} \sum_{k=1}^{s} P_k^r .$$

In the situation where the battle terminated after the shots of the first volley,

$$T_n \simeq T_o \prod_{r=1}^{r=n} \prod_{k=0}^{k=s} (1-\frac{1}{T_o} P_k^r).$$

This result was further simplified to

$$T_n \simeq T_o \exp \left\{ -\frac{1}{T_o} \sum_{r=1}^{n} \sum_{k=0}^{s} p_n^k \right\}$$

under conditions which permit approximation of a binonial random variable by a Poisson random variable (this was called the

single volley Poisson approximation). Two final approximations made by Helmbold were the expected value approximation and its Poisson counterpart. In the former, the value of $t_n$ was approximated by its conditional expectation, which could then be solved iteratively to yield

$$T_{n+1} \simeq T_n \prod_{k=1}^{s} \left(1 - \frac{1}{T_n} p_k^n\right).$$

The result obtained in the Poisson expected value approach was that

$$T_{n+1} \simeq T_n \exp \left\{ - \frac{1}{T_n} \sum_{k=1}^{s} p_k^n \right\} .$$

Helmbold made an extensive comparative analysis of the above approximations, from which he concluded that the expected value approximation yielded results which were closest to the exact values for the widest range of parameter choices.

Robertson [48] also considered a continuous analog to her original model in which h was taken as the time interval between firings. The number of shots fired per volley, s, equaled wrh, where w is the number of active weapons in the force and r is the number of shots fired per weapon per unit time. She showed that T'(t), the derivative of the number of targets still alive at time t, could be approximated by $-wrp(t)T(t)\frac{1}{T_o}$, where $p(t) = \frac{1}{s} \sum_{k=1}^{s} P_k^n$. Helmbold's modification simplified this to $T'(t) = -wrp(t)$.

Dolansky [18] cited that Brown [13] had worked on the problem of determining the probability of winning in a stochastic Lanchester-based model of warfare. Brown showed that

$P(r,b)$, the probability that the red side wins, equaled $A(r,b)P(r,b-1) + [1-A(r,b)]P(r-1,b)$, where $A(r,b)$ is the probability that at any given time the next casualty will occur on the red side. As is so often the case, the precise solution for $P(r,b)$ was difficult to compute. Thus, in addition to specifying its exact form for both the linear law (when $A(r,b)$ is a constant) and the square law (when $A(r,b) = er(er+b)$ and e is a ratio of relative effectiveness of red to blue), Brown derived several approximate solutions.

A recent stochastic formulation by Marshall [41] treated the problem of permitting each side to receive replacements. It will be recalled that one of Lanchester's simplifying assumptions was that a force would never increase in strength as the battle progressed. Marshall formulated the problem as a two-dimensional random walk with boundaries. He assumed that when $r=R$, the red side could never receive replacements (similarly when $b=B$) and that combat ended with b winning when $r = 0$, or with r winning when $b = 0$. At any instant in time, the point $(r,b)$ could do any one of five things: it could remain unchanged or it could go to $(r+1,b)$, $(r,b+1)$, $(r-1,b)$, or $(r,b-1)$. When the battle was of considerable length and transitions between states occurred with considerable frequency, the discrete changes in strength could be approximated by two simultaneous differential equations which Marshall derived. His equations were:

$$\frac{dR}{dt} = -pb + m(r-R),$$

and

$$\frac{dB}{dt} = -gr + n(b-B).$$

In these equations, m is the resupply rate for the red side per unit of red per unit time (similarly for n), p is the attrition rate against r per unit of b per unit time (similarly for q). Marshall concluded his work by determining the duration of the battle in the case where $m = n$, $p = q$, and $R = B$.

A summary article by Wallis [57] discussed several of the stochastic approaches already mentioned, and attempted to categorize those situations when such an approach was reasonable. In his view, there were three general settings when a stochastic formulation was appropriate. The first such setting was when there were relatively few units engaged in battle, which was "likely when the battle is bloody". Secondly, high correlation between losses indicated that a deterministic formulation of the problem could not give accurate results. Thirdly, modeling the combat stochastically was suggested when the payoff desired by the decision-maker was dependent upon the form of the probability distribution of the survivors.

Wallis praised the stochastic formulation for the freedom it permitted in the choice of the probability density function for each of the attrition coefficients, but cautioned, as have so many others, that when the deterministic approach is discarded, it must be remembered that computational

difficulties grow rapidly as the sizes of the opposing forces grow.

The most recent contribution to a stochastic theory of combat found in the research done for this paper was a 1970 article by Koopman [38]. He pointed out that the analysis of any combat situation must include consideration of three essential ingredients: a means of thoroughly describing the system, an array which specifies all of the possible states into which the system may be driven, and a means for identifying the role played by the military decision-maker in affecting the transitions from state to state. Koopman was led to a stochastic formulation of combat by early recognition of the fact that given a set of input parameters, seldom will two plays of any battle proceed exactly the same way. He described combat as a stationary Markov stochastic process and justified his assumptions in a manner not unlike authors of earlier articles. In cautioning about the use of the Markov assumption, he noted that such an assumption is unjustified in a situation where, when a particular state (r,b) has been achieved at time t, some factor is still not completely known and can be better understood by knowledge of some earlier state of the system. Several methods for solution of the stochastic equations were described; among these techniques were separation of variables, Green's function, infinite series, successive approximations, and approximation by difference equations (replacing continuous time t by relatively short time periods of length $\Delta t$). Also outlined

in the article were two ways of establishing the relationship between Lanchester's deterministic equations and the stochastic Lanchester equations. The first recalled that from a stochastic viewpoint, the strengths of each side at a given instant are random variables, whereas in the original Lanchester formulation these numbers represented expected values. In the second approach, the differences in the stochastic formulation could be replaced by their derivative approximations. In an effort to shed light on the enormity of the calculations often required in the stochastic analysis, Koopman noted that the number of states possible for a given system grows large very rapidly, particularly when it is recognized that a target must, in general, be detected before it can be destroyed. Thus, the number of equations which must be written down to completely describe the system might be so great as to deter one from an exploration of a stochastic analysis.

In light of the difficult calculations required by a general stochastic formulation, Morse and Kimball [43] and Willard [63] demonstrated that the solution to the original deterministic Lanchester equations was analogous to the expected outcome in a stochastic formulation. They concluded that a deterministic approach was appropriate. Brooks [11] argued in favor of stochastic determinism.

## D. OPERATIONAL LOSSES, REPLACEMENTS, AND MOVEMENT OF FORCES

Lanchester's original equations described attrition due to hostile fire only. Non-combat losses were first examined

by Morse [42] and subsequently by Bach, et. al. [2]. The latter article considered how large an initial strength a side would need to insure victory, and how a winning side must act in order to minimize its total losses. To consider operational losses, the authors rewrote Lanchester's equations as follows:

$$\frac{dR}{dt} = -A_{br}B(t) - kR(t),$$

and

$$\frac{dB}{dt} = -A_{rb}R(t) - fB(t),$$

where k is the operational loss rate for the red side (similarly for f). The condition on initial strength needed to win was shown to be

$$\frac{r_o^{\,2}}{b_o^{\,2}} > \frac{A_{br}}{A_{rb}} - \frac{(k-f)}{A_{rb}} \left(\frac{r_o}{b_o}\right),$$

which of course simplifies when k = f. It was also shown that a winner can reduce its total losses by committing to the battle a greater number of men than the minimum required for victory, provided $c < 1$, where

$$c = \frac{\frac{1}{2}(k+f)}{[A_{rb}A_{br} + \frac{1}{4}(f-k)]^{\frac{1}{2}}} .$$

In a manner analogous to that described above except with the sign of the second term positive instead of negative, Marshall [41] modeled combat which permitted each side to accept reinforcements.

In a 1957 article, Weiss [59] revised the Lanchester square law formulation to account for the fact that the rate at which two sides will lose men is critically dependent on how far apart the forces are during an engagement. He included a term, g(s), which was a function of the force separation distance. Thus,

$$\frac{dR}{dt} = -A_{br}B(t)g(s).$$

If $s_r$ is the distance of the red force from some reference line, and $s_b$ is the distance of the blue force from the same line, then $s = s_r - s_b$. Noting that $ds/dt = ds_r/dt - ds_b/dt$, Weiss argued that since movement of each force relative to the other is affected by the casualties each side has suffered, retreat and advancement could be predicted by his model. He quite reasonably assumed that each commander has, generally by order of higher headquarters, a certain casualty rate he is willing to accept. With this assumption, a comparison of observed casualty rates with acceptable rates would dictate whether or not a force should continue to advance or begin to retreat.

Gamow and Zimmerman [23] also considered movement of forces. However, their model did not consider that the attrition rate coefficients $A_{br}$ and $A_{rb}$ are affected by the distance which separates the two sides.

A slightly different approach to movement of forces and range-varying kill rates was made by Bonder [4]. His form

of the square law incorporated force separation distance, s, into the attrition rate coefficient. His equations were:

$$\frac{dR}{dt} = -A_{br}(s)B(t),$$

and

$$\frac{dB}{dt} = -A_{rb}(s)R(t),$$

indicating the coefficients as functions of s. Noting that $\frac{dR}{dt} = \frac{dR}{ds}\ \frac{ds}{dt}$, and that $\frac{ds}{dt} = v$, he rewrote the above equations as:

$$v\frac{dR}{ds} = -A_{br}(s)B(t)$$

and

$$v\frac{dB}{ds} = -A_{rb}(s)R(t).$$

Then, differentiating with respect to s, noting

$$B(t) = -\frac{v}{A_{br}(s)}\ \frac{dR}{ds},\ \frac{dB}{ds} = -\frac{A_{rb}(s)}{v}\ R(t),$$

and

$$\frac{dv}{ds} = \frac{dv}{dt}\cdot\frac{1}{v},$$

and finally substituting these expressions in the resulting derivative and dividing by v, he came up with the following two equations:

$$\frac{d^2R}{ds^2} + \frac{dR}{ds}\left\{\frac{a}{v^2} - \frac{1}{A_{br}(s)}\ \frac{d}{ds}\ [A_{br}(s)]\right\} - \left[\frac{A_{br}(s)A_{rb}(s)}{v^2}\right]R$$
$$= 0$$

and

$$\frac{d^2B}{ds^2} + \frac{dB}{ds}\left\{\frac{a}{v^2} - \frac{1}{A_{rb}(s)}\frac{d}{ds}[A_{rb}(s)]\right\} - \left[\frac{A_{br}(s)A_{rb}(s)}{v^2}\right]B$$

$$= 0 ,$$

where a is the acceleration between forces (so that

$$\frac{dv}{dt} \cdot \frac{1}{v} = \frac{a}{v}) .$$

Clearly, these two final expressions showed that a relationship exists between casualties and force separation and movement.

In the same article, Bonder discussed an armor battle in which tanks of the blue side attacked a static red position at constant speed. He assumed a constant linear kill rate, implying that

$$A_{br}(s) = k_{A_{br}(s)}\ (R_e - s) ,$$

where $R_e$ is the maximum effective range of blue's tanks; i.e., $A_{br}(s) = 0$ if $s > R_e$. By letting $X = (R_e - s)^2$ and using chain rule differentiation, he was able to reduce his second-order differential attrition equations to a solvable form. The solutions to these new equations showed that greater mobility of the attacking tanks reduced their losses in the engagement, but also reduced the number of hits by the blue tanks on the static red force.

## E. VERIFICATION STUDIES

A great number of authors who have looked at Lanchester's theory over the years have pointed out that only limited work

has been done to compare actual casualty data with predictions made by application of Lanchester's equations. As Dolansky [18] stated, there is a need for more "...verification studies in order to establish the validity of Lanchester-type equations more firmly in more sophisticated situations."

Perhaps the best-known verification study was done by Engel [19] in 1954, in which he showed that actual casualties suffered by both the United States and Japan during the seizure of Iwo Jima compared very favorably with casualty figures as predicted by Lanchester's equations. (Engel's model included a term which allowed for reinforcements.) A similar result was obtained by Weiss [59] when he looked at the battles on several Pacific islands during World War II and concluded that the use of Lanchester's square law was justified for casualties on the United States side.

The most exhaustive verification study to date was done by Willard [63]. It was significant primarily for two reasons. In the first place, Willard looked at more than 1500 land battles which had occurred during the period 1618-1905, so he certainly had a large sample space from which to draw his conclusions. Secondly, his skeptical conclusions about the applicability of Lanchester's equations were startling. Willard placed every battle in one of two categories, meeting engagements or attacks on fortified positions. His results demonstrated that while data from attacks on fortified positions showed a higher correlation to Lanchester's predictions than did the data from meeting engagements, in neither case

was the correlation strong enough to allow one to conclude that Lanchester's equations accurately describe attrition in combat. In his view, the accuracy of Lanchester's equations deteriorated rapidly as the size of the initial forces on each side increased, until, for very large battles, Lanchester's approach simply could not be used. Willard further concluded from his in-depth study that in a fight to the finish, a winner could not be predicted by examination of the initial force ratio. While this opinion contrasted sharply with the work of many others, Helmbold [28] supported Willard's claim in a study of 92 land battles done in 1961. While Helmbold was uncertain about the full range of factors which led to victory on the battlefield, he was convinced at least that numerical superiority was not the only such factor. From Lanchester's square law, Helmbold estimated the ratio of attrition coefficients, $A_{br}/A_{rb}$, from the data he had, and plotted ln $A_{br}/A_{rb}$ versus ln $R_o/B_o$. This scatter diagram indicated the existence of a linear regression. He let $V = \ln u$ (where $u^2 = A_{br}R_o{}^2/A_{rb}B_o{}^2$), since a linear relation between ln $A_{br}/A_{rb}$ and ln $R_o/B_o$ will maintain itself as a linear relation between V and ln $R_o/B_o$. Utilizing the techniques of linear regression, he determined the regression equation to be $V = b + c\ln R_o/B_o$, where b and c are constants. He went on to show that V was an index of the blue side's probability of winning. More importantly, his research indicated that $A_{br}/A_{rb}$ was positively correlated with $R_o/B_o$, rather than negatively correlated as one would expect if numerical superiority in a battle was an indicator of victory.

A verification analysis which looked at battles of the U. S. Civil War was made by Weiss [58]. As Willard had done before him, he separated the battles into two categories. He termed the first category "assaults on fortified lines" and grouped all remaining confrontations under the heading of "other battles". Before a discussion of the results of the Weiss study, the implications of Lanchester's original formulations are deserving of mention. Under Lanchester's linear law, the casualty rate was not dependent upon initial force ratio, while under the square law the casualty rate was inversely proportional to the force ratio. For the battles which could be classified as attacks on fortified positions, Weiss showed that the attackers' losses were directly proportional to the number of defenders trying to repel the attack, but that wide battle-to-battle variability existed in casualty ratios. He further concluded that the probability of the attackers winning the battle increased as the ratio of attackers to defenders increased.

In the category of "other battles", into which 28 of the Civil War encounters were placed, Weiss was noncommital about the dependence (or lack thereof) of casualty ratios on force ratios. He could not prove such dependence, but nor could he disprove it. To be an effective prediction device, he concluded that any combat model must be "... based on the ability to continue fighting as a function of sustained fractional losses." He developed such a model for the probability of winning. The formulation of this model indicated that the

initial force ratio was a strong determinant of which side would win. For the battles which he examined, casualty ratios were shown to be independent of which side attacked. In general, the losing side suffered 15 percent casualties as compared with 12 percent on the winning side.

A recent study which looked at historical data from Korea was based on the most complete information of any such study to date [22]. The authors, Fain, et. al., were equipped with the following data on a daily basis from each of three battles: the amount of close air support, the amounts of heavy and light artillery, the friendly and enemy strengths, the friendly and enemy casualties, and the relative movement of the foward edge of the battle area. In their model, the tactical warfare simulation program, they used two methods for computing kill rates. In the empirical method, kill rates were derived from an analysis of known engagement times, strengths, and casualties suffered. In the theoretical method, attrition was calculated from a term called the index of firepower potential, which measured the ability of a unit to inflict casualties on its opponents. It was assumed in the model that casualty production took the form described by Lanchester's equations; that is, for close combat, losses on each side were proportional to the strength of the opposition, and for fire support, losses on the side receiving the fire were proportional to the product of the strengths of each side. The authors recognized the dependence of casualty production upon weapon composition, ammunition expenditure rates,

and the positioning of the forces of the target unit. For each of the three battles studied, the fit was good for both the empirical and the theoretical approaches, with the empirical method consistently giving better results. For example, for the Seoul-Inchon landing (15 September through 30 October 1950), actual casualties were 2534 and 19,142 for the Allies and the North Koreans, respectively. The empirical method yielded results of 2507 and 19,370, while the theoretical method computed losses of 2401 and 18,072. As should be clear from these figures, the article concluded that Lanchester's theory for prediction of casualties was valid for the data examined.

Additional verification studies using data from Korea have been made by Overholt [44] in which it was determined that Lanchester's equations worked well for predicting losses on the Chinese communist side, but not so well for United States' casualties. At the time of this writing, Overholt [44] and Low [40] were examining the applicability of Lanchester's equations to data from the war in Vietnam.

## F. INSURGENCY AND GUERRILLA WARFARE

In the years since the end of the Korean War, considerable effort has been undertaken by military planners to come up with a suitable strategy for combating insurgency and guerrilla warfare. Many have felt that this type conflict will dominate the scene for many years to come. Thus, it seems suitable at this point to examine the work which has

been done in applying Lanchester's equations to this special type of warfare.

Early efforts by Deitchman [17] to model guerrilla warfare deterministically are well known. A more recent insurgency model was developed by Shaffer [50]. He derived deterministic forms of Lanchester's equations for each of three types of encounter: the skirmish, the ambush, and the siege. He considered a large number of small (100-man) forces operating in an area. Neither side could receive reinforcements. Both sides had supporting weapons; the insurgents support came from small, portable weapons such as mortars and recoilless rifles, while the counterinsurgents had, in addition, ground-attack aircraft. The equations Shaffer derived were identical to Lanchester's original square law formulation except that each force was also reduced in size by the effects of the supporting weapons of its opponents. Thus, each equation contained the term

$$- \sum_i E_i(t) W_i \ ,$$

where $W_i$ is the number of supporting weapons of type i which the side has, and $E_i(t)$ is the effectiveness of one weapon of type i at time t. Because surrenders and desertions are common in insurgencies, Shaffer derived differential equations which permitted a commander to predict the losses each side would suffer from those sources. For the skirmish, where each side maneuvers against the other and surprise is not a factor, Shaffer considered the terms $E_i$ and $E_j$ as

constants. He then obtained a numerical solution to his revised equations on a computer. For the ambush, it was recognized that the force being ambushed reduced its exposure in time and thus also reduced the effectiveness of the ambushers' weapons. He obtained numerical solutions to the ambush equations which showed that the attacker would win, even though outnumbered by those he was ambushing, because of the initial cover advantage he possessed. Modeling of the siege (attack on a fortified position such as a strategic hamlet) required some modification. The siege is generally preceeded by an artillery and air preparation. Thus, in the model, Lanchester's linear law was applicable for that phase since the riflemen did not fire their weapons (out of range). During the assault phase, the square law was again applicable. Shaffer's assault phase in the siege model was identical (in the absence of defensive artillery support) to the model for fixed area defense developed earlier by Brackney [9]. Shaffer's solution for the assault phase was independent of time.

Subsequent research done by Shaffer [51] on guerrilla warfare included a development of additional models for skirmishes, ambushes, and sieges which were unique in that they used weapon efficiency coefficients which were time-dependent. While these models could not be used to predict a winner, they were valuable because they permitted critical analysis of casualty claims. He showed that in any guerrilla operation, morale and the state of discipline are important

determinants of how long a battle will last, who will win it, and the rate at which casualties will be inflicted. It was demonstrated for the skirmish that the time selected for the introduction of supporting weapons was critical. For ambushes, Shaffer predicted success against an opponent twice as large as oneself. In preparing for a siege, it was shown that a commander must weigh the pros and cons of using preparatory fires. If he used them, they were of assistance in "softening up" the enemy, but the effects of the element of surprise were thereby severely diminished, if not lost. Shaffer developed descriptive equations which could be used by the commander of the attacking force to help him decide whether or not to employ these preparatory fires.

In 1966, Kisi and Hirose [36] reformulated Deitchman's Lanchester-type model of guerrilla warfare in stochastic form. They considered a blue force of guerrillas waiting to ambush an approaching red force of regulars. When the blue force began firing, all the red force was in full view. On the other hand, at the start of the engagement, the red force could only fire into the area it suspected the guerrillas occupied. The authors considered a stochastic formulation of the problem to be appropriate because of the small number of combatants on the guerrilla side. The results of their research were formulas for determining the probability of winning, both exactly and approximately. In their derivation, p(r,b) was the probability that the red side wins and q(r,b) was the probability that the blue side wins (in the

exact derivation, these two quantities were assumed to sum to one). The time between shots fired by the guerrillas was taken to be exponentially distributed with mean $1/\lambda'$. The time between successive firings by the regulars had the same distribution with mean $1/\lambda$. The regulars' fire was uniformly distributed over an area, A, and $A_e$ was taken as the effective lethal area of a single shot, while r* and b* were the strengths which, when reached, would cause a side to disengage. The formula derived by Kisi and Hirose was:

$$p(r,b) = \sum_{j=r^*+1}^{j=r} \left[ \frac{(-1)^{r-j} j^{r-r^*-1}}{(r-j)!(j-r^*-1)!} \right] \left[ \frac{j}{j+\alpha} \right]^{b-b^*} ,$$

where

$$\alpha = \frac{\lambda' pA}{\lambda A_e} .$$

By dropping the assumption that $p(r,b) + q(r,b) = 1$, an approximate formula was derived for p(r,b) by the use of generating functions:

$$p(r,b) \simeq \sum_{j=b-b^*}^{\infty} \left( \frac{u^j}{j!} \right) e^{-u} ,$$

where

$$u = \frac{r^2 - r^{*2}}{2\alpha} .$$

To demonstrate the goodness of their approximation, Kisi and Hirose showed that for $\alpha = 500$ and $r^* = b^* = 0$, the exact value for p(100,100) was 0.5460, while the approximation yielded a result of 0.542, an error of less than one percent.

## G. THE PROBABILITY OF WINNING

Any commander would certainly welcome information about his chances of success prior to the start of any battle. Accordingly, the concept of the probability of winning has received considerable attention over the years by many who have looked at Lanchester's equations. Three efforts in this area have already been discussed in other contexts; specifically, Kisi and Hirose [36] in guerrilla warfare, Brown [12] in a stochastic formulation of combat and Helmbold [31] in his verification study (see above). A fresh look at the probability of winning was an extension of the work of Brackney [9] and Brown [13] which was done by Smith [52]. Brackney had described nine possible combat situations which could come about when two sides faced each other. Brown's work resulted in a recurrence formula that could be used to evaluate the probability of winning in five of these nine situations. Using the same assumptions that had been made by Brown, Smith was able to generate the probability distribution of the number of survivors of each side for all nine possible situations. From these distributions he determined by use of the calculus of finite differences, the probability that a given side would win. Additionally, he derived formulas for computing the probability that a given number of the red force would be alive when the blue force was annihilated (and vice versa) from knowledge of the initial strengths of each side.

## H. DUELS AND INDIRECT FIRE

In an extensive summary article on mathematical models which describe combat, Bonder [5] showed that the original Lanchester linear law describes two tactical situations: the duel and indirect fire into an area where one's enemy cannot be precisely located. His solution to the simultaneous differential equations which describe these two forms of conflict was simple and permitted rapid calculation of r(t) and b(t), the number of survivors on either side at time t, given initial strengths and the values of $A_{br}$ and $A_{rb}$.

A stochastic treatment of the duel was recently made by Hellman [25]. In his model he considered combat as a series of individual duels, comprising a process which is neither stationary nor Markovian. He hypothesized that the starting time of duels was a Poisson process with rate $\lambda$. The duration of a given duel was a random variable whose distribution might take on any of several forms (in his example the exponential distribution was assumed). He assumed that the termination of a duel occurred when one or both the parties involved were killed. Using basic probability theory he then developed probabilities of winning for each side and probabilities that there were $r_t$ and $b_t$ survivors on each side at time t.

## I. THE ROLE OF COMMAND EFFICIENCY AND INTELLIGENCE

Success in combat is, of course, determined by several factors which the original Lanchester equations either did not consider or erroneously assumed to be the same from

battle to battle. Among these factors are the value of good intelligence and the efficiency of the command. A very simple model which incorporated these two broad categories was developed by Schreiber [49], in which his assumptions closely paralleled the rather primitive ones made by Lanchester. The important difference considered by Schreiber was that he permitted each side during a battle to receive through its intelligence system information on the results of its fire against its opponents. With the use of this information, the command and control system of the unit could then order the subsequent fire to be redirected such that it was continually uniformly distributed over the surviving members of the other side. Clearly, the better the intelligence, the less fire would be wasted. Schreiber measured the effectiveness of the intelligence and command and control system by a term referred to as command efficiency (CE), and defined as "the fraction of the enemy's destroyed units from which fire has been redirected." A perfect system then was one in which CE = 1, while the worst case (most fire wasted) occurred when CE = 0. The equations which Schreiber derived were:

$$\frac{dR}{dt} = - \frac{A_{br}B(t)R(t)}{[r_o - CE_b(r_o - r)]} ,$$

and

$$\frac{dB}{dt} = - \frac{A_{rb}R(t)B(t)}{[b_o - CE_r(b_o - b)]} .$$

Note that when CE = 0 (no information), area fire resulted and when CE = 1 (total information), the situation was identical with aimed fire. The above equations have limited applicability because Schreiber assumed that each side remained stationary throughout the battle and because he assumed command efficiency does not diminish, as it must as each side suffers casualties. Nevertheless, Schreiber's results were important because they showed that a unit's fighting capability can be greatly influenced by the worth of its intelligence and command and control systems.

## J. THE ATTRITION COEFFICIENTS

As has been pointed out earlier, the attrition rate, $A_{br}$, is the constant rate at which a single blue unit kills red units. These coefficients are seldom easy to determine, for as Bonder [5] said, there is no way of "... predicting the attrition coefficients ... as a function of a weapon system's capabilities...." As mentioned by Peterson [45], most models specify $A_{br}$ and $A_{rb}$ as functions of the rate of fire of one side and the exposed area of targets of the other side (area fire) or of rate of fire and single shot kill probability (aimed fire). While Bonder [6] was unable to predict these coefficients, he did derive a probability distribution of attrition rate for weapons which were able to adjust their fire based on feedback information on the results of previously fired rounds. In this work, Bonder considered average attrition as equivalent to the arithmetic mean of a set of attrition rates. His use of an arithmetic mean was later

challenged by Barfoot [3] who attempted to justify the use of the harmonic mean instead. (Bonder [7] subsequently published another article demonstrating that he and Barfoot were really saying the same thing.) Barfoot's model did have, however, more general applicability than did Bonder's model. Bonder had considered a case where a weapon system had only three different hit probabilities and one conditional probability of a kill, whereas Barfoot permitted m different hit and kill probabilities for each weapon.

A computer method of solving the Lanchester-type differential equations was obtained by Dashiell and Fain [16]. In their study, $A_{br}$ and $A_{rb}$ were not considered continuous, but rather "temporarily constant" so that each could be viewed as a step function, constant for some short time interval of length $\Delta t$. The values of $A_{br}$ and $A_{rb}$ at the start of any new time interval were considered to be determined solely by the state of the battle at the end of the previous time interval. When either of the two coefficients changed in the tactical warfare simulation program which Dashiell and Fain used, that signalled the end of one time interval and the beginning of another. Their program determined values for $A_{br}$ and $A_{rb}$ and also computed lengths of time over which the coefficients so determined could be considered as constant.

With this background, Dashiell and Fain derived a series of ordinary differential equations for describing combat attrition. The excessive computer time necessary to solve these equations led the authors to an approximation by the use of difference equations:

$$\Delta B^j \approx -A_{rb} R^j \Delta t_j ,$$

and

$$\Delta R^j \approx -A_{br} B^j \Delta t_j ,$$

where $B^j$, for example, is the number of blue survivors at the start of the $j^{th}$ time period, and $\Delta t_j$ is the length of the $j^{th}$ time period. In their model, $\Delta t_j$ was taken to be "... the time period between changes in the engagement status of any unit, or one hour, whichever comes first." (Additional work on this tactical warfare simulation program was reported by Fain, Fair, and Karr [21].)

## K. ENGAGEMENTS PRECEEDED BY SEARCH AND DETECTION

All the studies examined thusfar modeled warfare which began when one side initiated fire against its opponent. However, since combat cannot begin until one force is detected by another, a model which considers the search and reconnaissance aspects of a battle can provide valuable information to a commander. The work of Brackney [9] was a significant contribution to many offshoots of Lanchester's theory, and his paper has been discussed by several authors in recent years. To avoid repetition of the results of research done by others, only Brackney's ideas on search will be discussed here. As was mentioned briefly earlier in this paper, he identified nine possible combat situations which could arise on the battlefield, depending upon the posture (attack, constant area defense, and constant density defense)

of each force. He defined $T_r$ (the time required by the red force to fire its weapons) to be the sum of two quantities, $T_{rs}$ and $T_{rf}$, where $T_{rs}$ is the time required for the red force to search for and detect the blue force, and $T_{rf}$ is the elapsed time between finding the enemy and firing at him. He assumed that detection time for the red force was inversely proportional to the density of the blue targets, or that $T_{rs} = k_r A_b/b$, where $A_b$ is the area the blue force occupies and k is a proportionality constant (a similar expression was specified for $T_{bs}$). With this background, Brackney described combat (under conditions where search was required prior to engagement) by the following equations:

$$\frac{dR}{dt} = - \frac{p_b{}^b}{\left[k_r\left(\frac{A_r}{r}\right) + T_{bf}\right]} ,$$

and

$$\frac{dB}{dt} = - \frac{p_r{}^r}{\left[k_r\left(\frac{A_b}{b}\right) + T_{rf}\right]} .$$

He then examined several special cases from which he concluded that when $T_{rs} >> T_{rf}$, the square law applied and a defensive posture should be assumed by the blue force, and when $T_{rs} << T_{rf}$, the linear law applied and an attack posture was indicated for the blue force. This showed that the linear law could sometimes be used even for aimed fire.

## L. FOLLOWING DOLANSKY'S SUGGESTIONS

At the end of his article, Dolansky [18] identified several problems for future research. Among these was the task of "development of outcome-predicting relations that use only information pertaining to friendly units, in the heterogeneous cases." In a recent work, Buell, et. al. [14] described how one might compute approximate values for enemy strength and enemy replacement rates of men and materiel from information about one's own side alone. The authors began with three differential equations of attrition. The first equation described enemy attrition in terms of friendly strengths and enemy replacement rates, while the second and third equations described attrition of friendly direct fire and indirect fire weapons, respectively. Using arbitrary initial conditions, these equations were then integrated to yield time histories of friendly forces. To determine time histories of enemy strengths and replacement rates, the process of quasilinearization (a quadratically converging successive approximation scheme) was used. Arbitrary but reasonable values were chosen for dR/dt and c (replacement rate). The next step involved linearization of the three Lanchester equations by expanding their right-hand sides in power series about the values selected arbitrarily, and retaining only the linear terms. Taking the duration of a battle to be unity, particular and complementary solutions to these linearized equations were then computed numerically on the interval zero to one. A constant multiplier of the

complementary solutions was also calculated. Four quantities were solved for by this method: enemy strength, enemy replacement rate, and two friendly strengths (direct and indirect fire weapon strengths). The process was repeated with new choices for the enemy figures until such time as the values computed by this procedure for the friendly strengths were close enough to the actual, observed friendly strengths. That solution which gave the best fit between friendly strengths (calculated versus observed) was then taken as the correct solution for enemy strength and replacement rate.

Dolansky also recommended that further study be undertaken in the area of "...optimum target assignment problems, using the differential-game theory approach." He referred to an effort by Isbell and Marlow [34] which dealt with the fire distribution problem by using a terminal control (one-sided) differential game. Their setting had a homogeneous red force opposed by a heterogeneous blue force of, for example, riflemen and machine gunners. The red commander's problem was to determine what proportion of his fire to allocate to each type weapon on the blue side. Isbell and Marlow did not completely solve this problem. The conditions under which the various terminal states of combat are reached were not determined. A complete solution to the problem has been derived by Taylor [54].

In an unpublished paper, Taylor [56] has used the theory of optimal control to establish that the form of attrition

process under consideration, the conditions which signalled the end of a battle, and the level of command efficiency of the combatants had a strong influence on the "tactics for target selection."

## M. SMALL COMBAT GROUPS AND WEAPONS WITH GREAT EFFECTIVENESS

In contrast to the large-scale battles considered by many authors in testing the applicability of Lanchester's equations, Weiss [59] modeled a conflict consisting of a series of battles between relatively small groups of combatants. He assumed that the commander of the red side divided his force into several equally-numbered groups of size $m_r$ (similarly for the blue side with groups of size $m_b$). When two groups met in battle, they fought until one side was annihilated. The survivors on the winning side were then reinforced up to full size again to seek out battle with another enemy group. The equations Weiss derived were:

$$\frac{dR}{dt} = - \frac{A_{br}B(t)R(t)}{m_r} ,$$

and

$$\frac{dB}{dt} = - \frac{A_{rb}R(t)B(t)}{m_b} .$$

Notice that these equations reduced to the linear law when $m_r = m_b = 1$, and to the square law when $m_r = R$ and $m_b = B$.

An extension by Weiss in the same article produced attrition equations for the circumstances under which each

side possessed air-deliverable weapons with devastating power. He assumed such weapons produced casualties proportional to the number of men in the group against which the weapon was delivered. Letting $X_R$ and $X_B$ be the number of high-yield weapons being used by the red and blue sides respectively, and letting $k_{rb}$ and $k_{br}$ be the attrition coefficients for these weapons, the equations derived were:

$$\frac{dR}{dt} = - \frac{A_{br}B(t)R(t)}{m_r} - k_{br}X_b m_r \quad ,$$

and

$$\frac{dB}{dt} = - \frac{A_{rb}R(t)B(t)}{m_b} - k_{rb}X_r m_b \quad .$$

N. FORCE SIZES GROSSLY UNEQUAL

One of the assumptions of the original Lanchester square law attrition model was that the rate at which casualties are produced is independent of the relative size of the two forces. Helmbold [26] took issue with this assumption by advancing the idea that if one force outnumbered the other by a considerable margin, say 40 or 50 to 1, it was unreasonable to assume that the larger force could bring all of its potential combat power to bear on its smaller enemy. He therefore suggested an alternative set of differential equations which accounted for large differences in force size. His equations were:

$$\frac{dR}{dt} = - A_{br}B(t)\,g\left(\frac{R(t)}{B(t)}\right) \quad ,$$

and

$$\frac{dB}{dt} = -A_{rb}R(t)h\left(\frac{B(t)}{R(t)}\right) ,$$

where g and h are functions which adjust the coefficients $A_{br}$ and $A_{rb}$ under gross force size differences ($g(1) = h(1) = 1$, so that the square law resulted where R and B were approximately equal). As an example, Helmbold let $z = R(t)/B(t)$ and assumed $h = z^c$, where c is a constant. He then solved the resulting differential equations and showed that they reduced to the square law with $c = 0$, and to the linear law with $c = \frac{1}{2}$.

## IV. COMMENTS

Early in this paper it was pointed out that Lanchester's description of combat was based on several simplifying assumptions. Weapon systems and tactics are so sophisticated and diverse today that Lanchester's original model cannot stand alone as a predictive tool for the analysis of ground combat because some of these assumptions (such as assuming homogeneous forces are facing each other) are no longer valid. Nevertheless, Lanchester's efforts represented a significant contribution to military operations research. The theory, with its revisions and extensions, has added a new dimension to the military decision maker's thought process. Without Lanchester's theory a commander makes decisions in combat by applying the principles of war (see Conolly, R. L., "The Principles of War," Proc. U. S. Naval Institute, Vol. 79, No. 1, p. 1-9 (1953)) and the tactical expertise he has gained through schooling and experience. Lanchester's theory may aid the military leader in at least three specific areas. First, it may provide a quantitative basis for some of the principles of war. For example, the principle of mass dictates that a commander concentrate the bulk of his forces at the decisive time and place in any battle. Lanchester demonstrated the advantages of concentration of forces in modern non-nuclear warfare. Secondly, the ideas conveyed by the principles of war may be supplemented by additional information gained from application of Lanchester's

theory, information predicting chances of success and expected outcomes of alternative courses of action. Thirdly, Lanchester's theory may provide insight into the dynamics of combat, how and why conditions may be expected to change in time on the battlefield.

No attempt was made at a critical analysis of the numerous ideas cited herein. This thesis was intended solely to array for the interested analyst the recent efforts that have been made in the modeling of combat with Lanchester's theory.

The future will doubtless bring new and more wide-ranging studies in Lanchester's theory. It is hoped that efforts will be directed in the following areas:

1. Verification studies using data from the war in Southeast Asia, such as those now being conducted by Low and Overholt.

2. Studies designed to determine the minimum force that can be expected to adequately accomplish a given mission. Work in this area should be initiated as rapidly as possible in view of the anticipated reductions in the size of our armed forces.

3. The development of less cumbersome solution techniques for the stochastic formulation of combat, and an expansion of the stochastic formulation to account for the wide variety of weapons expected to be available on battlefields of the future.

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13. ABSTRACT

Since its introduction more than half a century ago, Lanchester's theory of combat has undergone considerable revision and expansion. This thesis presents a consolidation and a grouping by subject of the significant contributions to this theory which have appeared in the literature in recent years, with emphasis on the period since 1964, when a thorough "state of the art" summary article was published by Dolansky. The important results of each notable effort are described.

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